THE DETERMINATION OF INTEGRAL CLOSURES AND GEOMETRIC APPLICATIONS

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Abstract. We express explicitly the integral closures of some ring extensions; this is done for all Bring-Jerrard extensions of any degree as well as for all general extensions of degree ≤ 5; so far such an explicit expression is known only for degree ≤ 3 extensions. As a geometric application we present explicitly the structure sheaf of every Bring-Jerrard covering space in terms of coefficients of the equation defining the covering; in particular, we show that a degree-3 morphism $\pi : Y \to X$ is quasi-etale if and only if $c_1(\pi_* \mathcal{O}_Y)$ is trivial (details in Theorem 5.3). We also try to get a geometric Galoisness criterion for an arbitrary degree-n finite morphism; this is successfully done when $n = 3$ and less satifactorily done when $n = 5$.

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Introduction

The computation of the integral closure (or normalization) of a finite extension is a fundamental problem in number theory, commutative algebra and algebraic geometry. David Hilbert's determination of the ring of algebraic invariants is exactly the computation of normalization (see [St]). Readers will also notice that Noether's normalization theorem appears as the basic pre-requisite of Algebra at page 3 of David Mumford's "The red book of varieties and schemes".

The computation of integral closure for cyclic extensions is well known (Lemma 1.3). For a cubic extension, this computation was pioneered by A. A. Albert in the 1930's [Al1, Al2], and continued and completed half century later in [Sp], [ShS] over **Z** and in [Ta1] over a Noetherian unique factorization domain (UFD for short). They extended a result of Richard Dedekind [De] published in 1899, who has given an integral basis for a pure cubic field. However, as far as the authors know, if the extension is non-cyclic with degree higher than 3, it seems that no general formula has been found.

For a general affine domain, important pioneering works have been done by W. V. Vasconcelos in [Va1] who established a very effective algorithm to compute the integral closure (see also [BV] and [dJ]). For the recent development, the analysis of the algorithm and its complexity and more background, we refer readers to Vasconcelos's very comprehensive book [Va2] as well [Va3] and [SUV]. The results in the book [Va2] are very important in simplifying our argument and make our description of integral closure possible.

A finite extension $R[\alpha]$ of a UFD R is given by a root α of an irreducible polynomial in $R[z]$:

$$f = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \quad a_i \in R.$$

By a linear Tschirnhaus transformation $z \mapsto z - a_{n-1}/n$, one can assume that the coefficient of z^{n-1} vanishes. Jerrard proved that by a Tschirnhaus transformation involving square and cube roots, the second, third and fourth terms (after the leading term) of a general

polynomial f can be removed. This result generalized Bring's result for quintic polynomials. (See [Dh, pp.195–200]).

To be precise, a Tschirnhaus transformation is the substitution $w = \alpha_0+\alpha_1 z+\cdots+\alpha_m z^m$ for some $m \le n-1$ and some α_i to be determined by solving square and cube equations with solutions in an over ring $\hat{R}$ of R, so that if $f(z) = 0$ then $g(w) = 0$ for some polynomial g in $\hat{R}[w]$ of the form

$$g = w^n + b_{n-4}w^{n-4} + \cdots + b_1 w + b_0.$$

In particular, when $n \le 5$, the determination of roots of a general degree n polynomial can be reduced to that of a polynomial of the following form

$$z^n + sz + t$$

after making one base change of degree 2 if $n = 4$ or some base changes of degree ≤ 3 if $n = 5$. Such a polynomial is called a *Bring-Jerrard polynomial* (or simply a B-J *polynomial*). The corresponding extension (with the polynomial irreducible) is called a *Bring-Jerrard extension* (or simply a B-J *extension*).

The purpose of this paper is to express explicitly the integral closure $\widetilde{A}$ of a Bring-Jerrard extension $A = R[\alpha]$ given by the root α of a degree n Bring-Jerrard polynomial over R (Theorem 2.1). As an application, we calculate explicitly the integral closure of a general degree 4 or 5 extension by reducing it to a type B-J extension, using Tschirnhaus transformation (Theorem 3.1 and Theorem 4.5).

We compute also the ramification divisor of a B-J extension (Theorem 1.4). Late, we apply the computation of integral closure to algebraic geometry; especially, we determine explicitly the structure sheaf of the covering space in terms of coefficients of the equation defining the covering (Theorem 5.1). As a further application, we prove a geometric criterion for a degree-3 finite morphism to be Galois (Theorem 5.5); the general degree$-n$ case is more complicated, and we prove a partial result for the case of degree-5 (Proposition 5.7).

The idea of our computation is as follows. The normalization ring $\widetilde{A}$ must be a reflexive R-module for any f [Ha2]. On the other hand, any reflexive R-module M of rank n is a syzygy module

$$0 \to M \to R^{m+n} \xrightarrow{\varphi_M} R^m$$

for some m, where φ_M is an $(m+n)\times m$ matrix over R. In order to find the integral closure $\widetilde{A}$ defined by f, we need to find the matrix $\varphi_f = \varphi_{\widetilde{A}}$ from f. This reduction has the advantage that the syzygy module $M = \widetilde{A}$ is always reflexive, and so satisfies automatically Serre's condition S_2 (see [Va1] or [Va2]). Hence we only need to compute the co-dimension one normalization. Serre's condition R_1 (co-dimension one nonsingular) is invariant under localization $R_{\mathfrak{p}}$ at all height-one prime ideals $\mathfrak{p}$ and under completion $\widehat{R}_{\mathfrak{p}}$. Hence we only need to compute the normalization of an extension over the one-dimensional ring $\widehat{R}_{\mathfrak{p}}$. By Cohen's Structure Theorem for regular rings [Ha1, p.34], $\widehat{R}_{\mathfrak{p}} \cong k[[x]]$ (assuming R contains a field). Thus we can assume that R is the ring of formal power series over the residue field k. The polynomial f defines a local curve in $\mathbf{A}^2_k$. Now the normalization is just the resolution of plane curve singularities, which can be realized by the embedded resolution [Ha1, p.391]. The next step is to globalize the local computation to R.

In fact, the syzygy presentation of the integral closure is the simplest one, because if R is a general UFD but not a PID, then there is no canonical method to solve the syzygy equations; namely, it is hopeless to give explicitly the generators of $\widetilde{A}$ when $n \ge 3$. Indeed, Akizuki [Re, §9.5] shows that there is a 1-dimensional Noetherian local ring R such that

the integral closure $\widetilde{R}$ of R in its field of fraction is not finitely generated as an R-module; in other word, the number of generators of $\widetilde{R}$ as an R-module is not finite, or the syzygy equations in the description of $\widetilde{R}$ can not be solved.

The result in this note is also related to the works of Catanese [Ca], Miranda [Mi], Casnati - Ekedahl [CE] and Hahn - Miranda [HM]. Compared with their results, our approach emphasizes more on the very close relation between the structure sheaf of the variety upstairs and the coefficients of the equation (downstairs) defining the finite morphism.

Assumption

For integral domain R we assume that $\mathrm{Char}\, R$ is coprime to both n and $n-1$, where n is the degree of the extension $R \subset R[\alpha]$.

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1. Discriminant and ramification of a Bring-Jerrard extension

Let $n \geq 3$ be an integer. In this section, we shall calculate explicitly the discriminant and ramification of the integral closure of a Bring-Jerrard extension $R \subset R[\alpha]$ of a Noetherian UFD R containing a field, which is defined by a root α of $f(z) = z^n + sz + t$ with non-zero elements s, t in R.

Following [Ta1], we shall decompose s and t as well as the discriminant of f as the products of elements defining reduced divisors in $\mathcal{S}pec\, R$. For a prime element p in R, we let $s_p = \nu_p(s)$ be the corresponding valuation of s. Now we define:

$$\varepsilon_p := nt_p - (n-1)s_p, \qquad \lambda_p := \min\left\{ \left[\frac{s_p}{n-1}\right], \left[\frac{t_p}{n}\right] \right\},$$
$$a_i := \prod_{\substack{\varepsilon_p > 0 \\ \varepsilon_p \equiv i\ (n)}} p, \qquad b_j := \prod_{\substack{\varepsilon_p < 0 \\ \varepsilon_p \equiv j\ (n-1)}} p,$$

where $1 \leq i \leq n-1$ and $1 \leq j \leq n-2$. If $\lambda_p > 0$ then $f/p^n = (z/p)^n + (s/p^{n-1})(z/p) + (t/p^n)$ is in $R[z]$ and we may replace f by f/p^n. Thus we may and will assume that $\lambda_p = 0$ for all prime element p, i.e., the data (s, t) is minimal (see **(1.1)** below). We can check easily the following decomposition:

$$s = a_0 \prod_{i=1}^{n-1} a_i^i \prod_{j=1}^{n-2} b_j^{n-1-j}, \quad t = b_0 \prod_{i=1}^{n-1} a_i^i \prod_{j=1}^{n-2} b_j^{n-j}.$$

The usual discriminant δ of $f(z)$ is given as follows (or the negative of it):

$$\begin{aligned} \delta &= (n-1)^{n-1} s^n - (-n)^n t^{n-1} \\ &= \left(\prod_{i=1}^{n-1} a_i^i \right)^{n-1} \left(\prod_{j=1}^{n-2} b_j^{n-1-j} \right)^n c, \end{aligned}$$

where

$$c := a + b$$

and

$$a := (n-1)^{n-1} a_0^n \prod_{i=1}^{n-1} a_i^i, \qquad b := -(-n)^n b_0^{n-1} \prod_{j=1}^{n-2} b_j^j.$$

Let $c_1 := \prod_{c_p=\text{odd}} p$. Then we can write $c = c_0^2 c_1$. From the definition of a_i, b_i and the relation $c = a + b$, we see easily:

- Since $s \neq 0$, these a, b and c are relatively coprime; if $i \geq 1$, then a_i, b_i and c_1 are square-free, i.e., they define reduced divisors of $\mathcal{S}pec\, R$;
- a_0, b_0 and c_0 are not necessarily square-free; and

$$\delta = \left(\prod_{i=1}^{n-1} {a_i}^i \right)^{n-1} \left(\prod_{j=1}^{n-2} {b_j}^{n-1-j} \right)^n c_1 c_0^2.$$

Definition 1.1. (1) $z^n + sz + t$ or the data (s, t) is called *minimal* if there is no prime divisor p such that $p^{n-1} \mid s$ and $p^n \mid t$, i.e., $\lambda_p = 0$ for all prime divisors p.

(2) $z^n + sz + t$ is said to be equivalent to $z^n + s'z + t'$ if there is a regular section e on X without zero point on X such that $s' = e^{n-1}s$ and $t' = e^n t$.

(3) The triplets (a, b, c) and (a', b', c') of coprime sections of a line bundle with $a + b = c$ and $a' + b' = c'$ are said to be equivalent if there is a regular section e without zero point on X such that $a' = ea$, $b' = eb$ and $c' = ec$.

If the data (s, t) is not minimal, i.e., if $\lambda = \prod_p p^{\lambda_p}$ is not a unit, then we can check easily that the defining polynomial of α/λ is a minimal B-J polynomial of degree n, and the normalization ring of $R[\alpha/\lambda]$ is equal to that of $R[\alpha]$. So we can always assume that the data (s, t) is minimal. Clearly, we have:

Proposition 1.2. *Let R be a Noetherian* UFD*. Then up to equivalences* **(1.1)***, there is a one to one correspondence between the following two sets:*

$$\{ \text{ Minimal } z^n + sz + t \text{ with } s \neq 0 \} \longleftrightarrow \{ \text{ Coprime } (a, b, c) \text{ with } a + b = c \}.$$

The following result is well-known and stated for later use (see [EV, pp.18–35]).

Lemma 1.3. *Let R be a Noetherian* UFD*, and let $A = R[\alpha]$ be a cyclic extension of R which is defined by a root α of an irreducible polynomial $z^n + u\ell_1\ell_2^2 \ldots \ell_n^n$ in $R[z]$, where $\ell_1, \cdots, \ell_{n-1}$ in R are square-free and u in R is a unit. Denote by $\widetilde{A}$ the integral closure of A. Then we have:*

(1) *As an R-module, $\widetilde{A}$ is generated by:*

$$\frac{\alpha^k}{\prod_{j=1}^n \ell_j^{[jk/n]}}, \quad k = 0, 1, \cdots, n-1.$$

(2) *Denote by $\mathfrak{p}$ a minimal ideal of R containing ℓ_k, and by $\widehat{R}_\mathfrak{p}$ the completion of the local ring $R_\mathfrak{p}$. Then the completion $\widetilde{A} \otimes_{R_\mathfrak{p}} \widehat{R}_\mathfrak{p}$ of $\widetilde{A}$ has exactly* $\gcd(n, k)$ *maximal ideals (all of height 1) whose generators $\bar{x}_{ki}$ can be chosen so that we have the following factorization:*

$$\ell_k = \left(\prod_{i=1}^{\gcd(n,k)} \bar{x}_{ki} \right)^{\frac{n}{\gcd(n,k)}}.$$

Now we can prove the main result of this section.

Theorem 1.4. *Let $n \geq 3$ be an integer. Let R be a Noetherian* UFD *containing a field with* $\mathrm{Char}\, R$ *coprime to n and $n-1$, and let $A = R[\alpha]$ be a $B-J$ extension of R given by a root α of an irreducible polynomial $f(z) = z^n + sz + t$ in $R[z]$ with $s \neq 0$ and the data (s,t) minimal (see* (1.1)*). Denote by $\widetilde{A}$ the integral closure of A. Then we have:*

(1) *The discriminant of the extension $\widetilde{A}$ over R (i.e., the defining equation of the branch locus of the finite cover $\mathcal{S}pec\, \widetilde{A} \to \mathcal{S}pec\, R$) is*

$$D_{\widetilde{A}/R} = c_1 \prod_{k=1}^{n-1} a_k^{n-\gcd(n,k)} \prod_{k=1}^{n-2} b_k^{n-1-\gcd(n-1,k)}$$

(2) *Let $\mathfrak{p}$ be a prime ideal of R generated by a factor x of a_k or b_k or c_k. Then we have the following factorizations into primes in the completion $\widetilde{A} \otimes_{R_\mathfrak{p}} \widehat{R}_\mathfrak{p}$ of $\widetilde{A}$,*

$$x = \begin{cases} \left(\prod_{i=1}^{\gcd(n,k)} \bar{x}_i\right)^{\frac{n}{\gcd(n,k)}}, & \text{if } \; x \,|\, a_k, \\ \left(\prod_{i=1}^{\gcd(n-1,k)} \bar{x}_i\right)^{\frac{n-1}{\gcd(n-1,k)}} \bar{x}'_1, & \text{if } \; x \,|\, b_k, \\ \bar{x}_1^2 \; \prod_{i=1}^{n-2} \bar{x}'_i, & \text{if } \; x \,|\, c_1. \end{cases}$$

In particular, we have the following factorizations of a_k, b_k and c_1 in $\widetilde{A}$, modulo the multiplication by a unit of $\widetilde{A}$, where $\bar{a}_k$, $\bar{b}_k$ and $\bar{c}_1$ $(k \geq 1)$ are square free

$$a_k = \bar{a}_k^{\,n/\gcd(n,k)}, \quad b_k = \bar{b}_k^{\,(n-1)/\gcd(n-1,k)}\, \bar{b}'_k, \quad c_1 = \bar{c}_1^2\, \bar{c}'_1.$$

(3) *The defining equation of the ramification divisor of the finite cover $\mathcal{S}pec\, \widetilde{A} \to \mathcal{S}pec\, R$ is*

$$\mathcal{R}_{\widetilde{A}/R} = \bar{c}_1 \prod_{k=1}^{n-1} \bar{a}_k^{\frac{n}{\gcd(n,k)}-1} \prod_{k=1}^{n-2} \bar{b}_k^{\frac{n-1}{\gcd(n-1,k)}-1}.$$

Proof. Note that the discriminant $D_{\widetilde{A}/R}$ is a factor of

$$\delta = D_{A/R} = (n-1)^{n-1}s^n - (-n)^n t^{n-1} = \left(\prod_{i=1}^{n-1} a_i{}^i\right)^{n-1} \left(\prod_{j=1}^{n-2} b_j{}^{n-1-j}\right)^n c_1 c_0^2.$$

So we need only to find the ramification indices over the prime ideal $\mathfrak{p} = (x)$ generated by a factor x of a_k $(k \geq 1)$, b_k $(k \geq 1)$ or c_k.

Localizing R at $\mathfrak{p}$, we may assume that $R = R_\mathfrak{p}$, which is a DVR, so that x is a parameter. Denote by $\mathfrak{m}$ a (height 1) minimal ideal of $\widetilde{A}$ over $\mathfrak{p}$. Since we consider only the ramification at $\mathfrak{m}$ over $\mathfrak{p}$, we may reduce to the completion $\widehat{R}$ with respect to $\mathfrak{p}$. So we may assume that R is complete. By Cohen's Structure Theorem [Ha1, p.34], a complete regular local ring of dimension 1 containing some field is the ring of formal power series over the residue field k, i.e., $R \cong k[[x]]$. Hence A is a local curve over k defined by $z^n + s(x)z + t(x) = 0$ in $\mathbf{A}_k^2$. $\widetilde{A}$ is the normalization of this curve. The ramification index of $\widetilde{A}$ at $\mathfrak{m}$ over $\mathfrak{p}$ is equal to the corresponding ramification index as a local n-cover over $\mathcal{S}pec\,(k[[x]])$. Since the normalization can be realized by the embedded resolution of a plane curve singular point, in what follows, we shall compute the ramification index by using Lemma 1.3 and the embedded resolution.

Note that $\widetilde{A}$ is a finite module over R, so $\widetilde{A}$ is also complete.

We consider first the case $x \mid a_k$. We rewrite the polynomial:

$$f(z) = z^n + a'x^m z + b'x^k = z^n + (b' + a'x^{m-k}z)x^k = z^n + ux^k,$$

where $m \geq k$, $u = b' + a'x^{m-k}z$, and a' and b' are units in R. Since $f(\alpha) = 0$ and x is in $\mathfrak{m}$, we see that $z = \alpha$ is in $\mathfrak{m}$. Thus u is a unit of $\widetilde{A}$. So we are reduced to Lemma 1.3. This proves Theorem 1.4 for factors of a_k.

Next we consider the case $x \mid b_{n-1-k}$. So

$$f = z^n + a'x^k z + b'x^m$$

where $m \geq k+1$, and a' and b' are units in R. Now we blow up $\mathbf{A}_k^2$ at $(0,0)$. Then the strict transform of the curve $C : f(z) = 0$, is locally a union of an irreducible component isomorphic to the original curve C and a curve (of degree $n-1$ over C) given below, where z and $x' = x/z$ are new coordinates and the divisor $x' = 0$ is also the strict transform of the divisor $x = 0$ on C

$$z^{n-k-1} + {x'}^k v = 0, \quad v := a' + b'{x'}^{m-k} z^{m-k-1}$$

Since this v is a unit, this case is reduced to Lemma 1.3 in applying which we note also that the total transform in $\mathcal{S}pec\, \widetilde{A}$ of the divisor $x = 0$ in $\mathcal{S}pec\, A$ is defined by $zx' = 0$. This proves Theorem 1.4 for factors of b_{n-1-k}.

Finally, we consider the case $x \mid c_k$ for $k = 0$ or 1. In this case, s and t are units of R. Set $w := z + \frac{nt}{(n-1)s}$. It is easy to know that if there is a ramification over $x = 0$, then we must have $w = 0$. We rewrite the polynomial

$$g(w) := f = w^n + \sum_{i=0}^{n-1} e_i w^i,$$

where

$$e_i = \begin{cases} -\dfrac{t\delta}{((n-1)s)^n}, & i = 0, \\ \dfrac{\delta}{((n-1)s)^{n-1}}, & i = 1, \\ C_n^i \left(\dfrac{-nt}{(n-1)s} \right)^{n-i}, & i \geq 2. \end{cases}$$

Hence e_i is a unit when $i \geq 2$. Clearly there is an $m \geq 1$ such that the following are all units in R

$$\delta/x^m, \ c_0^2 c_1/x^m, \ e_i/x^m \ (i = 0, 1).$$

We rewrite $g(w)$ as follows, where a', b' are units of R and u_1 and u_2 are units of $\widetilde{A}$.

$$\begin{aligned} g(w) &= w^n + \sum_{i=2}^{n-1} c_i w^i + a'x^m w + b'x^m \\ &= w^2 u_1 + x^m u_2. \end{aligned}$$

We see easily that if $x \mid c_1$ then m is odd and the ramification index is 2; if x does not divide c_1 (and hence x divides c_0) then the normalization has no ramification. This proves Theorem 1.4 for factors of c_k. The proof of Theorem 1.4 is completed.

2. Integral closure of a B-J extension

In this section, we will calculate explicitly the integral closure of a B-J extension.

Let $n \geq 3$ be an integer. Let R be a Noetherian UFD with $\mathrm{Char}\, R$ coprime to n and $n-1$, and let $A = R[\alpha]$ be a B-J extension of R defined by a root α of an irreducible polynomial $f(z) = z^n + sz + t$ in $R[z]$ with $s \neq 0$. Suppose that the data (s, t) is minimal, i.e., $\lambda_p = 0$ for all prime divisors p; see **(1.1)**. Denote by $\widetilde{A}$ the integral closure of A, and by $\widetilde{A}_0$ the trace-free R-submodule of $\widetilde{A}$. Obviously the trace map $\mathrm{Tr}: \widetilde{A} \to R$ splits because $\mathrm{Char}\, R$ is coprime with n (and also $n-1$) by the assumption in the Introduction. Thus

$$\widetilde{A} = R \oplus \widetilde{A}_0.$$

For $1 \leq i \leq n-1$, we set

$$h_i := \prod_{k=1}^{n-1} a_k^{[ki/n]} \prod_{k=1}^{n-2} b_k^{[(n-1-k)i/(n-1)]}.$$

If we denote by σ_i the coefficient of z^{n-i} in f, then we can compute $s_i = \mathrm{Tr}(\alpha^i)$ by Newton's identities:

$$\begin{aligned} &s_1 + \sigma_1 = 0, \\ &s_2 + s_1\sigma_1 + 2\sigma_2 = 0, \\ &\qquad \vdots \\ &s_k + s_{k-1}\sigma_1 + \cdots + s_1\sigma_{k-1} + k\sigma_k, \quad k \leq n. \end{aligned}$$

In our case, $\sigma_i = 0$ for $i < n-1$, $\sigma_{n-1} = s$ and $\sigma_n = t$. Thus we get

$$\mathrm{Tr}(\alpha^i) = 0, \text{ for } i = 1, \cdots, n-2, \quad \text{and } \mathrm{Tr}(\alpha^{n-1}) = -(n-1)s.$$

So we can construct $n-1$ trace free elements:

$$\beta_i := \begin{cases} \dfrac{\alpha^i}{h_i}, & \text{if } \ i = 1, \cdots, n-2, \\ \dfrac{\alpha^{n-1} + \frac{n-1}{n}s}{h_{n-1}}, & \text{if } \ i = n-1. \end{cases}$$

For $i = 1, \cdots, n-2$, we define

$$f_i := (n-1)a_0 \prod_{k=1}^{n-1} a_k^{\left[\frac{(i+1)k}{n}\right] - \left[\frac{ik}{n}\right]},$$

$$g_i := nb_0 \prod_{k=1}^{n-2} b_k^{1 + \left[\frac{(n-1-k)i}{n-1}\right] - \left[\frac{(n-1-k)(i+1)}{n-1}\right]}.$$

We now state the main result of this section.

Theorem 2.1. *Let $n \geq 3$ be an integer. Let R be a Noetherian* UFD *containing a field with* $\mathrm{Char}\, R$ *coprime to n and $n-1$ and let $A = R[\alpha]$ be a $B-J$ extension given by a root α of an irreducible polynomial $f(z) = z^n + sz + t$ in $R[z]$ with $s \neq 0$ and the data (s,t) minimal. Then the integral closure $\widetilde{A}$ of A in the fraction field of A satisfies $\widetilde{A} = R \oplus \widetilde{A}_0$, as R-modules, where the trace (over R) free R-submodule $\widetilde{A}_0$ of $\widetilde{A}$ is given as follows:*

$$\widetilde{A}_0 = \left\{ \frac{v_1\beta_1 + v_2\beta_2 + \cdots + v_{n-1}\beta_{n-1}}{c_0} \;\middle|\; v_j \in R,\ c_0 \mid f_k v_k + g_k v_{k+1},\ 1 \leq k \leq n-2 \right\}.$$

Corollary 2.2. *With the assumptions in* (2.1), $\widetilde{A}_0$ *is the following syzygy module*

$$0 \to \widetilde{A}_0 \to R^{2n-3} \xrightarrow{M} R^{n-2}.$$

Here M *sends* $(v_1, \cdots, v_{2n-3})$ *to* $(f_1v_1 + g_1v_2 + c_0v_n, f_2v_2 + g_2v_3 + c_0v_{n+1}, \cdots, f_{n-2}v_{n-2} + g_{n-2}v_{n-1} + c_0v_{2n-3})$.

Proof of Theorem 2.1. Note first that the $\widetilde{A}$ and $\widetilde{A}_0$ are reflexive [Ha2]. On the other hand, the right hand side of the displayed equality in the theorem is a syzygy module, so it is also reflexive. In order to get the equality, we only need to prove that the syzygy is the co-dimension one normalization of A. Because being reflexive implies the S_2 condition in Serre's Criterion [Ha1, p.185], or two reflexive modules over R are isomorphic if and only if they are co-dimension 1 isomorphic. Clearly, the right hand side of the displayed equality in the theorem, added with the summand R, contains $R[\alpha]$.

Now we use the same technique as in the proof of Theorem 1.4. Namely, we reduce the proof to the case when $R = k[[x]]$. Hence $A = k[[x,z]]/(z^n + s(x)z + t(x))$. Then we only need to prove that the normalization ring of the local curve singularity defined by $f = z^n + s(x)z + t(x) = 0$ can be generated by the syzygies in the theorem. Clearly, the theorem is true outside the set $\delta = 0$. So we may assume that $x \mid \delta$ and only need to check the equality in the theorem at the smooth points of $\mathrm{Supp}(\delta = 0)$.

If $x \mid a_k$, then as §1, we have $f = z^n + ux^k$, where u is a unit. Using Lemma 1.3, we can see easily that the normalization ring is generated by β_i. Hence it is generated by the syzygies.

If $x \mid b_{n-1-k}$, then
$$f = z^n + a'x^kz + b'x^m$$
where $m \geq k+1$, and a' and b' are units in R.

We shall prove first that these β_i are integral over R. Since the localization $\widetilde{A}_{\mathfrak{m}}$ of $\widetilde{A}$ at a height-1 prime ideal $\mathfrak{m}$ over (x) is a DVR, we can define a valuation $\nu = \nu_{\mathfrak{m}}$. Thus for any two elements g and h in $\widetilde{A}$, the element $r = g/h$ is in $\widetilde{A}_{\mathfrak{m}}$ if and only if $\nu(r) = \nu(g) - \nu(h) \geq 0$. Now we claim that $\nu(\beta_i) \geq 0$ for all i.

Suppose the contrary that $\nu(\beta_i) < 0$ for some i. If $i < n-1$, then we have

$$i\nu(\alpha) < \nu(h_i) = \left[\frac{ki}{n-1}\right]\nu(x) \leq \frac{ki}{n-1}\nu(x),$$

so $\nu(\alpha^{n-1}) = (n-1)\nu(\alpha) < k\nu(x)$ and $\nu(\alpha^n) < \nu(x^k\alpha)$. This and the equation $\alpha^n + a'x^k\alpha + b'x^m = 0$ imply that $\nu(\alpha^n)$ must be equal to $\nu(x^m)$. Hence

$$\frac{m}{n}\nu(x) = \nu(\alpha) < \frac{k}{n-1}\nu(x).$$

This implies that $\frac{n}{n-1} > \frac{m}{k} \geq \frac{k+1}{k}$. Hence we have $k > n-1$, a contradiction. Thus $i = n-1$. Then we have

$$\nu\left(\alpha^{n-1} + \frac{n-1}{n}s\right) = \nu\left(\alpha^{n-1} + \frac{n-1}{n}a'x^k\right) < \nu(h_{n-1}) = \nu(x^k) = \nu(s),$$

so $\nu(\alpha^{n-1}) < \nu(s) = k\nu(x)$, a contradiction as above. Hence β_i must be integral over R for all i.

Now we need to prove that these β_i generate the normalization ring. It is enough to prove that the discriminant $d = \det(\mathrm{Tr}(\beta_i\beta_j))$ of $\beta_0 = 1, \beta_1, \cdots, \beta_{n-1}$ is equal to the

discriminant $D_{\widetilde{A}/R}$ (see §1). From the definition of β_i, we get the following, where we set $h_0 := 1$ and replace β_{n-1} by α^{n-1}/h_{n-1}

$$d = \det\left(\mathrm{Tr}\left(\frac{\alpha^i\alpha^j}{h_ih_j}\right)\right) = \frac{\det(\mathrm{Tr}(\alpha^i\alpha^j))}{\prod_{i=1}^{n-1} h_i^2} = \frac{\delta}{\prod_{i=1}^{n-1} h_i^2}.$$

The identity $d = D_{\widetilde{A}/R}$ (up to a unit) is equivalent to

$$2\sum_{i=1}^{n}\left[\frac{ki}{n}\right] = (n+1)k - n + \gcd(n,k),$$

or

$$\frac{2}{n}\sum_{i=1}^{n}\varepsilon_i - n + \gcd(n,k) = 0,$$

where $0 < k < n$ and $\varepsilon_i = ki - n[ki/n]$ is a non negative integer less than n. Note that if $\gcd(n,k) = 1$, then ε_i attains all of the numbers between 0 and $n-1$, thus it is easy to obtain the above identity. The proof for the general case can be reduced to this case. So we have proved that these β_i are the generators of the normalization ring.

Finally we consider the difficult case when $x \mid c_1c_0^2$. As in §1, we rewrite $f(z)$ as follows, where e is a unit near $w = 0$

$$f(z) = w^2e + we_1 + e_0.$$

Since $c_0^2c_1 \mid e_i$ $(i = 0, 1)$ (§1), we see from the above equation that we/c_0 is integral over $A = R[\alpha]$, so it is also integral over R.

We shall find u_i in R such that

$$\beta' := \frac{we}{c_0} - \frac{1}{n}\mathrm{Tr}\left(\frac{we}{c_0}\right) = \frac{u_1\beta_1 + \cdots + u_{n-1}\beta_{n-1}}{c_0}.$$

If we let $q = -nt/(n-1)s$, then

$$we = \sum_{i=1}^{n-1} q^{n-1-i}z^i - (n-1)q^{n-1}.$$

Thus it is easy to see that

$$u_i = q^{n-1-i}h_i.$$

Now we claim that $\beta = \sum_{i=1}^{n-1} v_i\beta_i/c_0$ is integral over R if v_i satisfies the conditions in the expression of $\widetilde{A}_0$ in the theorem. In fact, one can check that these conditions are equivalent to the following

$$c_0 \mid u_{i+1}v_i - u_iv_{i+1}, \quad i = 1, \cdots, n-2.$$

In particular, β' satisfies the conditions. We consider the element $\beta'' = u_1\beta - v_1\beta' = (\sum_{i=1}^{n-1} v_i''\beta_i)/c_0$, $v_1'' = 0$. By using the induction on i and the above conditions, we see that v_i'' is divided by c_0. Hence β'' is integral over R. Since u_1 is a unit, β is also integral over R.

Next we need to prove that the syzygies generate the normalization ring $\widetilde{A}$ as an R-module. In fact, we only need to prove that $\beta^* = we/c_0$ generates the normalization ring near $w = 0$ (see §1). We shall show that $R[\alpha, \beta^*] = R[w, \beta^*] = R + R\beta^*$ is normal

near $x = w = 0$. As in §1, let $m \geq 1$ such that $c_0^2 c_1/x^m, e_i/x^m$ $(i = 0, 1)$ are units in R near $x = 0$. Here m is odd if and only if $x \,|\, c_1$. If $m = 1$ then the equation $f(z) = w^2 e + w e_1 + e_0 = 0$ has a linear term (near $x = w = 0$) and hence $R[\alpha] = R[z]/(f(z))$ is smooth near $w = 0$. We may assume $m \geq 2$ and hence $m \,|\, c_0$ for c_1 is reduced. Note that $R[w, \beta^*]$ is the quotient of $R[w, y] = k[[x]][w, y]$ modulo the equations below, where e is a unit in R near $w = 0$

$$F_1 := w - y c_0/e = 0, \quad F_2 := y^2 + \frac{e_1}{c_0} y + \frac{e_0}{c_0^2} e = 0.$$

If $x \,|\, c_1$ then $F_2 = 0$ implies that $y = 0$ (when $x = 0$) and F_2 has a linear term, whence $R[w, \beta^*]$ is smooth near $x = w = 0$. If x does not divide c_1, then the partial derivative $(F_2)_y \neq 0$ holds near $x = w = 0$ and along the zero locus of $F_2 = 0$, so is the smoothness of $R[w, \beta^*]$ near $x = w = 0$. We have completed the proof of Theorem 2.1.

3. Integral closure of a quartic extension

In this section, we will calculate explicitly the integral closure of a quartic extension $R[\alpha]$ of a Noetherian UFD R with $\operatorname{Char} R$ coprime to 2 and 3, which is given by a root α of an irreducible quartic monic polynomial $f(z)$ over R. We may assume, after a shift of coordinate, that

$$f(z) = z^4 + \sigma_2 z^2 - \sigma_3 z + \sigma_4.$$

We do the factorization:

$$2\sigma_2^3 - 8\sigma_2\sigma_4 + 9\sigma_3^2 = d_1 d_0^2,$$

where d_1 is square free in R, i.e., no square of a prime element of R divides d_1.

Consider the general case where d_1 has no square root in the fraction field of $R[\alpha]$. So we have ring extensions $R \subset R[\alpha] \subset R[\alpha, y]$ of degrees 4 and 2, where $y^2 = d_1$. Thus the ring extensions $R \subset R[y] \subset R[\alpha, y]$ are of degrees 2 and 4. In the new ring $\hat{R} := R[y]$, we can find an element

$$w = \frac{1}{2}\sigma_2^2 + \left(\frac{3}{2}\sigma_3 + \frac{1}{2} d_0 y\right)\alpha + \sigma_2 \alpha^2.$$

This $\hat{w} = w$ satisfies

$$\hat{w}^4 + \hat{s}\hat{w} + \hat{t} = 0,$$

where

$$\begin{aligned}
\hat{s} = & -\frac{1}{2}\sigma_2^6 + 4\sigma_2^4\sigma_4 - \frac{19}{4}\sigma_2^3\sigma_3^2 - 8\sigma_2^2\sigma_4^2 + 27\sigma_2\sigma_3^2\sigma_4 - \frac{27}{2}\sigma_3^4 \\
& - \frac{1}{4} y d_0 \sigma_3 (3\sigma_2^3 - 28\sigma_2\sigma_4 + 18\sigma_3^2), \\
\hat{t} = & \frac{19}{8}\sigma_2^5\sigma_3^2 - \frac{5}{4}\sigma_2^6\sigma_4 + \sigma_2^4\sigma_4^2 + \frac{3}{16}\sigma_2^8 + \frac{3}{2}\sigma_2^3\sigma_4\sigma_3^2 + \frac{81}{2}\sigma_4\sigma_3^4 \\
& + \frac{27}{4}\sigma_2^2\sigma_3^4 - 36\sigma_4^2\sigma_3^2\sigma_2 + 4\sigma_2^2\sigma_4^3 \\
& + y d_0 \left(\frac{3}{8}\sigma_2^5\sigma_3 + \frac{27}{2}\sigma_4\sigma_3^3 - 6\sigma_4^2\sigma_3\sigma_2 + \frac{9}{4}\sigma_2^2\sigma_3^3\right).
\end{aligned}$$

We consider the general case that the above degree-4 polynomial is irreducible over the fraction field of $\hat{R}$. Then we have ring extensions $R \subset \hat{R} \subset \hat{R}[w]$ of degrees 2 and 4. Since $\hat{R}[w] \subseteq R[y, \alpha]$ and both rings are of degree-8 extensions of R, they have the same fraction field; in particular, they have the same integral closure A^* in their common fraction field

$Q(R)[\alpha, y]$. We denote by A_0^* the $\hat{R}$-submodule of A^* consisting of elements of trace zero over $\hat{R}$. Since R is UFD and d_1 is also square free in R, the ring $\hat{R}$ is a normal ring, and thus co-dimension-one regular (see [Ma, §9, Example 4, p. 65]). We can factor any element in $\hat{R}$ into the product of primes over the smooth locus of $\hat{R}$. We do this for $\hat{s}$ and $\hat{t}$; note that $\hat{a}_i, \hat{b}_j$ below may have different expressions at different affine open sets of $\mathcal{S}pec\, \hat{R}$, and $\mathrm{div}(\hat{a}_i)$ and $\mathrm{div}(\hat{b}_i)$ are reduced divisors for all $i \geq 1$

$$\hat{s} = \hat{a}_0\, \hat{a}_1\, \hat{a}_2^2\, \hat{a}_3^3\, \hat{b}_1^2\, \hat{b}_2, \quad \hat{t} = \hat{b}_0\, \hat{a}_1\, \hat{a}_2^2\, \hat{a}_3^3\, \hat{b}_1^3\, \hat{b}_2^2.$$

We can also define $\hat{c}_k, \hat{f}_k, \hat{g}_k, \hat{h}_k, \beta_k, \cdots$ over the smooth locus of $\mathcal{S}pec\, \hat{R}$ as in §2. By Theorem 2.1, A_0^* is given as follows (over the smooth locus of $\mathcal{S}pec\, \hat{R}$):

$$A_0^* = \left\{ \frac{\hat{v}_1\beta_1 + \hat{v}_2\beta_2 + \hat{v}_3\beta_3}{\hat{c}_0} \;\middle|\; \hat{v}_j \in \hat{R},\ \hat{c}_0 \mid \hat{f}_k\hat{v}_k + \hat{g}_k\hat{v}_{k+1},\ k = 1,\ 2 \right\}.$$

We denote by $\bar{\bullet}$ the involution of $R[\alpha][y]$ over $R[\alpha]$, i.e., $\overline{r_0 + r_1 y} = r_0 - r_1 y$ for r_i in $R[\alpha]$.

Let $\widetilde{A}$ be the integral closure of $R[\alpha]$ in its fraction field and let $\widetilde{A}_0$ be the trace (over R) free part. Note that the following 6 elements are integral over R, contained in the fraction field of $R[\alpha]$ (for being the involution $\bar{\bullet}$-invariant) and trace (over R) free (noting that $\mathrm{Tr}|R[\alpha, y]/R[y]$ is the lifting of $\mathrm{Tr}|R[\alpha]/R$)

$$\gamma_i = \frac{\beta_i + \bar{\beta}_i}{2}, \quad \gamma_{i+3} = \frac{\beta_i - \bar{\beta}_i}{2} y, \quad i = 1, 2, 3.$$

So these 6 elements are in $\widetilde{A}_0$. Note that

$$\beta_i = \gamma_i + \gamma_{i+3}\frac{1}{d_1} y, \quad i = 1, 2, 3.$$

By the reasoning above, we have:

$$\widetilde{A}_0 = \left\{ \frac{\beta + \bar{\beta}}{2} \;\middle|\; \beta \in A_0^* \right\}.$$

Let

$$\begin{aligned}
&\hat{f}_k = f_k + f_{k+2} y, \quad \hat{g}_k = g_k + g_{k+2} y, \quad k = 1, 2;\\
&\hat{v}_i = v_i + v_{i+3} y, \quad i = 1, 2, 3;\\
&\hat{c}_0 = c_0 + c_0' y.
\end{aligned}$$

Then the elements of $\widetilde{A}_0$ can be expressed as

$$\sum_{i=1}^{6} v_i \mu_i,$$

where

$$\begin{aligned}
\mu_1 &:= c_0\, \gamma_1 - c_0'\, \gamma_4,\\
\mu_2 &:= c_0\, \gamma_2 - c_0'\, \gamma_5,\\
\mu_3 &:= c_0\, \gamma_3 - c_0'\, \gamma_6,\\
\mu_4 &:= -c_0'\, d_1\, \gamma_1 + c_0\, \gamma_4,\\
\mu_5 &:= -c_0'\, d_1\, \gamma_2 + c_0\, \gamma_5,\\
\mu_6 &:= -c_0'\, d_1\, \gamma_3 + c_0\, \gamma_6.
\end{aligned}$$

The 2 syzygies

$$\hat{f}_1\hat{v}_1 + \hat{g}_1\hat{v}_2 + \hat{c}_0(v_7 + v_9 y) = 0,$$
$$\hat{f}_2\hat{v}_2 + \hat{g}_2\hat{v}_3 + \hat{c}_0(v_8 + v_{10} y) = 0$$

induce 4 syzygies $\phi_i = 0,\ i = 1,\ 2,\ 3,\ 4$, where ϕ_i are defined by

$$\begin{aligned}
\phi_1 &:= f_1\, v_1 + g_1\, v_2 + d_1\, f_3\, v_4 + d_1\, g_3\, v_5 + c_0\, v_7 + d_1\, c_0'\, v_9,\\
\phi_2 &:= f_3\, v_1 + g_3\, v_2 + f_1\, v_4 + g_1\, v_5 + c_0'\, v_7 + c_0\, v_9,\\
\phi_3 &:= f_2\, v_2 + g_2\, v_3 + d_1\, f_4\, v_5 + d_1\, g_4\, v_6 + c_0\, v_8 + d_1\, c_0'\, v_{10},\\
\phi_4 &:= f_4\, v_2 + g_4\, v_3 + f_2\, v_5 + g_2\, v_6 + c_0'\, v_8 + c_0\, v_{10}.
\end{aligned}$$

Next we shall show that there are relations among μ_i. Indeed, by the generating property of $\beta_1, \beta_2, \beta_3$, we have

$$(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) = (\beta_1, \beta_2, \beta_3)M/\hat{c}_0,$$

where M is a 3×3 matrix with entries in $\hat{R}$. Expressing $\beta_i, \bar{\beta}_i$ in terms of γ_j we obtain the relation

$$(\gamma_1, \gamma_2, \gamma_3)(\hat{c}_0 I - M) = (\gamma_4, \gamma_5, \gamma_6)(\hat{c}_0 I + M)/y.$$

On the other hand, the definition of μ_i implies the following, where $|\hat{c}_0|^2 = \hat{c}_0\bar{\hat{c}}_0$

$$|\hat{c}_0|^2(\gamma_1, \gamma_2, \gamma_3) = c_0(\mu_1, \mu_2, \mu_3) + c_0'(\mu_4, \mu_5, \mu_6),$$
$$|\hat{c}_0|^2(\gamma_4, \gamma_5, \gamma_6) = d_1 c_0'(\mu_1, \mu_2, \mu_3) + c_0(\mu_4, \mu_5, \mu_6).$$

So the relation among γ_i implies a relation among μ_i:

$$(\mu_1, \mu_2, \mu_3)[\bar{\hat{c}}_0 I - M] = (\mu_4, \mu_5, \mu_6)[\bar{\hat{c}}_0 I + M]/y.$$

The main theorem below of this section follows from the arguments above and the fact that an element in a normal ring $\widetilde{A}$ is determined by its restriction to the open set lying over the smooth locus of $\mathcal{S}pec\, R$ (whose complement has co-dimension at least two).

Theorem 3.1. *Let R be a Noetherian* UFD *containing a field with* $\mathrm{Char}\, R$ *coprime to* 2 *and* 3. *For a general degree-*4 *extension $R \subset R[\alpha]$, where α is a root of an irreducible polynomial $f(z) = z^4 + \sigma_2 z^2 - \sigma_3 z + \sigma_4$ in $R[z]$, the integral closure $\widetilde{A}$ of $R[\alpha]$ in the fraction field of $R[\alpha]$ is given by $\widetilde{A} = R \oplus \widetilde{A}_0$, and the trace* (*over R*) *free R-submodule $\widetilde{A}_0$ of $\widetilde{A}$ is given as follows:*

$$\widetilde{A}_0 = \left\{ \sum_{i=1}^{6} v_i\mu_i \;\middle|\; \begin{array}{c} v_1, \cdots, v_{10} \text{ in } R \textit{ form a solution of the} \\ \textit{linear equations } \phi_k = 0,\ k = 1, 2, 3, 4 \end{array} \right\}.$$

Remark 3.2. (1) In Theorem 3.1, by a general degree-4 extension, we mean that $\tau := 2\sigma_2^4 - 8\sigma_2\sigma_4 + 9\sigma_3^2$ has no square root in R (so assume it is written as $d_0 d_1 = d_0 y^2$ with d_1 square free in R) and that $\hat{w}^4 + \hat{s}\hat{w} + \hat{t}$ defined above is irreducible over the fraction field of $R[y]$; this also implies that y is not in the fraction field of $R[\alpha]$, because the extension $R \subset R[y, w]$ is of degree 8, whence the extension $R \subset R[y, \alpha]$ is also of degree 8 and both extensions share the same fraction field.

(2) If the τ above has a square root in R then the extension $R \subset R[\alpha]$ is very likely to be of type B-J (at least when $\hat{w}^4 + \hat{s}\hat{w} + \hat{t}$ is irreducible as a polynomial over R). For B-J extension, we refer to Theorem 2.1.

4. Integral closure of a general degree-n extension

In this section, we will calculate explicitly the integral closure of a quintic extension $R[\alpha]$ of a Noetherian UFD R with $\mathrm{Char}\, R$ coprime to $5, 3, 2$, which is given by a root α of an irreducible quintic monic polynomial $f(z)$ over R.

We remark that the case of a general degree-n extension is similar, though the computation will be more complicated and the simpler polynomial (we may possibly reduce to) is of the form $z^n + a_{n-4}z^{n-4} + \cdots + a_1 z + a_0$. We will illustrate by considering the case of degree $n = 5$.

One may assume, after a shift of coordinate, that

$$f(z) = z^5 + \sigma_2 z^3 - \sigma_3 z^2 + \sigma_4 z - \sigma_5.$$

(4.1) The base change to reduce to type B-J case. We shall show that for a suitably general $f(z)$ we can find

$$y := u + v\alpha + w\alpha^2 + p\alpha^3 + q\alpha^4,$$

where u, v, w, p, q are in a (relatively not so big) over ring $\hat{R}$ of R, such that $\hat{y} = y$ is a zero of the following type B-J polynomial with coefficients in $\hat{R}$:

$$\hat{y}^5 + \hat{s}\hat{y} + \hat{t}.$$

In other words, after base changes, the extension $R \subset R[\alpha]$ may be reduced to the extension $\hat{R} \subset \hat{R}[y]$ of B-J type. We may then apply Theorem 2.1 to get the integral closure of $\hat{R}[y]$ and also that of $R[\alpha]$ in their respective fraction fields.

Let Z denote the matrix representation of the R-linear map

$$\alpha : (1, z, z^2, z^3, z^4) \mapsto (z, z^2, z^3, z^4, z^5 = -(\sigma_2 z^3 - \sigma_3 z^2 + \sigma_4 z - \sigma_5)).$$

Then the matrix representation Y of the linear map y is given by $Y = uI + vZ + wZ^2 + pZ^3 + qZ^4$. Note that y is a zero of the characteristic polynomial of Y:

$$|\lambda I - Y| = \lambda^5 + \sum_{i=0}^{4} d_i \lambda^i.$$

Here d_i is a homogeneous polynomial over R of degree $5 - i$ in u, v, w, p, q. We want to find u, v, w, p, q in some over ring $\hat{R}$ of R such that $d_i = 0$ for all $i = 2, 3, 4$. Then we just set $\hat{s} = d_1$ and $\hat{t} = d_0$ and get the desired y satisfying a B-J equation with coefficients in $\hat{R}$.

Step 1. Solve $d_4 = 0$. We get the following expression of u, which will be substituted to all of d_i:

$$u = \frac{1}{5}(-3p\sigma_3 + 2w\sigma_2 + 4q\sigma_4 - 2q\sigma_2^2).$$

Step 2. Note that d_3 is a quadratic form in v, w, p, q. We find the standard normal form of d_3:

$$d_3 = \mu_1\lambda_1^2 - \mu_2\lambda_2^2 + \mu_3\lambda_3^2 - \mu_4\lambda_4^2,$$

where μ_i are elements in R and λ_j are linear forms in v, w, p, q with coefficients in R. Write $\mu_2/\mu_1 = \mu_{2,1}(\mu'_{2,1})^2$ such that $\mu_{2,1}$ is square free in R and $\mu'_{2,1}$ is in the fraction field of

R. Then the extension $\hat{R}_1 := R[\sqrt{\mu_{2,1}}]$ of R is a normal ring, since R is UFD (see [Ma, §9, Example 4, p. 65]). So the singular locus of $\mathcal{S}pec\,\hat{R}_1$ is of co-dimension at least two, outside of which $\hat{R}_1$ is regular and hence a UFD.

Over the smooth locus of $\mathcal{S}pec\,\hat{R}_1$, we write $\mu_4/\mu_3 = \mu_{4,3}(\mu'_{4,3})^2$ where $\mu_{4,3}$ is a square free regular function of $\mathcal{S}pec\,\hat{R}_1$ and $\mu'_{4,3}$ is in the fraction field of $\hat{R}_1$. Then the extension $\hat{R}_2 := \hat{R}_1[\sqrt{\mu_{4,3}}]$ of $\hat{R}_1$ is a normal ring in the open set of $\mathcal{S}pec\,\hat{R}_2$ lying over the smooth locus of $\mathcal{S}pec\,\hat{R}_1$. Hence the singular locus of $\mathcal{S}pec\,\hat{R}_2$ is of co-dimension at least two.

Now any solution (v, w, p, q) satisfying linear equations in v, w, p, q below will also satisfies the equation $d_3 = 0$

$$\lambda_1 - \sqrt{\mu_{2,1}}\mu'_{2,1}\lambda_2 = 0, \quad \lambda_3 - \sqrt{\mu_{4,3}}\mu'_{4,3}\lambda_4 = 0.$$

Step 3. Note that d_2 is a cubic form in v, w, p, q. We substitute the two linear equations in Step 2 into d_2. Then d_2 will become a cubic form in only two of the 4 variables v, w, p, q, say in w, q only. Consider the general case that the Galois group of the cubic polynomial d_2/q^3 in w/q over the fraction field of $\hat{R}_2$ is equal to S_3. Taking a linear transformation of coordinates (w, q) over $\hat{R}_2$, we may assume that d_2 multiplied by some non-zero elements in $\hat{R}_2$, is equal to the following cubic form in (w, q) over $\hat{R}_2$ (where the new w and q are $\hat{R}_2$-linear combination of the old w and q)

$$w^3 + s_1wq^2 + t_1q^3.$$

As in §1, if $s_1 \neq 0$, over the smooth locus of $\mathcal{S}pec\,\hat{R}_2$, we write $s_1 = a_{10}a_{11}a_{12}^2b_{11}, t_1 = b_{10}a_{11}a_{12}^2b_{11}^2$ and the discriminant $4s_1^3 + 27t_1^2 = (a_{11}a_{12}^2)^2b_{11}^3c_{11}c_{10}^2$. Let $\hat{R}_3 := \hat{R}_2[\sqrt{b_{11}c_{11}}]$, which is normal over the smooth locus of $\mathcal{S}pec\,\hat{R}_2$, so the singular locus of $\mathcal{S}pec\,\hat{R}_3$ is of co-dimension at least 2.

Suppose that (w, q) is a zero of the cubic form above. We now define γ (set $\gamma = w/q$ if $s_1 = 0$):

$$\gamma := 6a_{10}(w/q)^2 - 9b_{10}b_{11}(w/q) + \sqrt{3}c_{10}\sqrt{b_{11}c_{11}}(w/q) + 4a_{10}^2a_{11}a_{12}^2b_{11}.$$

Then, using the fact that $c_{11}c_{10}^2 = 4a_{11}a_{12}^2a_{10}^3 + 27b_{11}b_{10}^2$, one can check that γ satisfies an equation

$$\gamma^3 = \ell_1\ell_2^2\ell_3^3,$$

where $\operatorname{div}(\ell_i)$ $(i = 1, 2)$ are reduced divisors of $\mathcal{S}pec\,\hat{R}_3$ and ℓ_3 is in $\hat{R}_3$. Since $\hat{R}_3[\gamma] \subset \hat{R}_3[w/q]$ and since these two rings have the same degree over $\hat{R}_3$, they have the same fraction field (= the splitting field of d_2/q in w/q over $\hat{R}_2$) and the same normalization. We define

$$\hat{R} = \hat{R}_3 + \hat{R}_3\gamma_1 + \hat{R}_3\gamma_2,$$

$$\gamma_k = \frac{\gamma^k}{\prod_{j=1}^3 \ell_j^{[jk/3]}}.$$

By Lemma 1.3, $\hat{R}$ is a rank-3 free $\hat{R}_3$-module and coincides with the normalization of the ring $\hat{R}_3[\gamma]$ (or equivalently of the ring $\hat{R}[w/q]$) on its open set lying over the smooth locus

of $\mathcal{S}pec\,\hat{R}_3$. So the singular locus of $\mathcal{S}pec\,\hat{R}$ is of co-dimension at least 2. Also for some u_k in $\hat{R}_3$, we have the following, since w/q is integral over $\hat{R}_2$ (and also over $\hat{R}_3$)

$$\frac{w}{q} = \sum_{k=0}^{2} u_k \gamma_k.$$

Step 4. Using the above linear relation, the two linear equations at the end of Step 2 and the linear equation in Step 1, we can write

$$u = u_1 q, \;\; v = v_1 q, \;\; w = w_1 q, \;\; p = p_1 q$$

such that these 4 coefficients of q are in the fraction field $Q(\hat{R})$ of $\hat{R}$ and that $y := u + v\alpha + w\alpha^2 + p\alpha^3 + q\alpha^4$ satisfies a type B-J equation : $\hat{y}^5 + \hat{s}\hat{y} + \hat{t} = 0$, where $\hat{s} = d_1 = \hat{s}_1 q^4, \hat{t} = d_0 = \hat{t}_1 q^5$ with coefficients of q^4, q^5 in $Q(\hat{R})$. Replacing y, $\hat{s}$, $\hat{t}$ by their multiples of elements in $\hat{R}$, we may assume that y is in $\hat{R}[\alpha]$ already and satisfies a B-J equation defined over $\hat{R}$.

(4.2). Here are some detailed calculations which are followed by an example. In Step 2 above, if $\sigma_2 \neq 0$ and if $\tau_i \neq 0$ $(i = 1, 2)$, where

$$\begin{aligned}
\tau_1 =& 45\sigma_3^2 + 12\sigma_2^3 - 40\sigma_2\sigma_4,\\
\tau_2 =& 160\sigma_4^3 + 117\sigma_4\sigma_3^2\sigma_2 + 12\sigma_2^4\sigma_4 - 88\sigma_2^2\sigma_4^2 - 4\sigma_2^3\sigma_3^2 +\\
& (-27)\sigma_3^4 + 125\sigma_2\sigma_5^2 - 40\sigma_2^2\sigma_3\sigma_5 - 300\sigma_4\sigma_3\sigma_5,
\end{aligned}$$

then we can write $d_3 = V - W + P - Q$, where

$$\begin{aligned}
V =& \sigma_2 \left(v - \frac{1}{2\sigma_2}\left(5q\sigma_5 + 2p\sigma_2^2 - 4p\sigma_4 + 3w\sigma_3 - 5q\sigma_3\sigma_2\right)\right)^2,\\
W =& \frac{\tau_1}{20\sigma_2}\left(w - \frac{1}{\tau_1}W'\right)^2,\\
W' =& 60p\sigma_4\sigma_3 + 8\sigma_2^2 p\sigma_3 - 75q\sigma_3\sigma_5 + 45q\sigma_3^2\sigma_2 +\\
& (-50)p\sigma_2\sigma_5 + 12\sigma_2^4 q - 44\sigma_2^2 q\sigma_4,\\
P =& \frac{\tau_2}{\tau_1}\left(p - \frac{q}{2\tau_2}P'\right)^2,\\
P' =& 195\sigma_3^2\sigma_2\sigma_5 - 375\sigma_3\sigma_5^2 + 36\sigma_2^4\sigma_5 - 4\sigma_4\sigma_3\sigma_2^3 +\\
& 48\sigma_4^2\sigma_3\sigma_2 + 400\sigma_5\sigma_4^2 - 260\sigma_2^2\sigma_5\sigma_4 - 27\sigma_3^3\sigma_4,\\
Q =& \left(\frac{1}{20\tau_1\tau_2}Q' - 4\sigma_3^2\sigma_2 + 4\sigma_3\sigma_5 - \frac{12}{5}\sigma_4\sigma_2^2 + \frac{2}{5}\sigma_4^2 + \frac{3}{5}\sigma_2^4\right)q^2.
\end{aligned}$$

Here Q' is a homogeneous polynomial of degree 26 over $\mathbf{Z}$ in σ_i, where we set $\deg(\sigma_i) = i$. In notation of Step 2,

$$\begin{aligned}
\frac{\mu_2}{\mu_1} &= \frac{\tau_1}{20\sigma_2^2},\\
\frac{\mu_4}{\mu_3} &= \frac{1}{20\tau_2^2}\left(Q' + 20\tau_1\tau_2\left(-4\sigma_3^2\sigma_2 + 4\sigma_3\sigma_5 - \frac{12}{5}\sigma_4\sigma_2^2 + \frac{2}{5}\sigma_4^2 + \frac{3}{5}\sigma_2^4\right)\right).
\end{aligned}$$

Example 4.3. We choose σ_i below so that μ_{i+1}/μ_i in **(4.2)** above are relatively simpler:

$$\sigma_2 = \frac{3}{10}, \quad \sigma_3 = \frac{1}{150}, \quad \sigma_4 = \frac{21}{2000}, \quad \sigma_5 = \frac{-427}{75000}.$$

Then the linear equations mentioned in Step 4 above are:

$$\begin{aligned} u &= \frac{1}{5}\left(-3p\sigma_3 + 2w\sigma_2 + 4q\sigma_4 - 2q\sigma_2^2\right), \\ v &= \frac{1}{9000}\left(654p + 3300w - 1463q\right), \\ p &= \frac{1}{6}q, \\ L &= 0. \end{aligned}$$

Here L is a linear factor of the cubic form below (which is d_2 multiplied by $2^4 \times 3^7 \times 5^{11}$)

$$612630271q^3 + 900w\left(-2004300qw - 2283643q^2 + 7590000w^2\right).$$

(4.4). Now we shall calculate the integral closure of $R[\alpha]$. Consider the general case where the polynomial $\hat{y}^5 + \hat{s}\hat{y} + \hat{t} \in \hat{R}[\alpha]$ found in **(4.1)**, is irreducible over the fraction field of $\hat{R}$. Let $\hat{y} = y \in R[\alpha]$ be a zero of this polynomial as found in **(4.1)**.

Set $\hat{A} := \hat{R}[y]$. Then $\hat{A}$ is a degree-5 extension of $\hat{R}$. Since the extension $\hat{R}[\alpha]$ of $\hat{R}$ contains $\hat{A}$ and $z = \alpha$ satisfies the degree-5 polynomial $f(z)$ in $R[z] \subset \hat{R}[z]$, these two extensions of $\hat{R}$ have the same fraction field (and $f(z)$ is irreducible over the fraction field of $\hat{R}$) and hence the same normalization, which we denote by A^*.

We denote by A_0^* the $\hat{R}$-submodule of A^* consisting of elements of trace zero over $\hat{R}$. We will factor any element in $\hat{R}$ into the product of primes over the smooth locus of $\mathcal{S}pec\,\hat{R}$. We do this for $\hat{s}$ and $\hat{t}$, the coefficients of the B-J polynomial in **(4.1)**. Note that $\hat{a}_i, \hat{b}_j$ below may have different expressions at different affine open sets of $\mathcal{S}pec\,\hat{R}$, and $\mathrm{div}(\hat{a}_i)$ and $\mathrm{div}(\hat{b}_i)$ are reduced divisors for all $i \geq 1$

$$\hat{s} = \hat{a}_0\,\hat{a}_1\,\hat{a}_2^2\,\hat{a}_3^3\,\hat{a}_4^4\,\hat{b}_1^3\,\hat{b}_2^2\,\hat{b}_3, \quad \hat{t} = \hat{b}_0\,\hat{a}_1\,\hat{a}_2^2\,\hat{a}_3^3\,\hat{a}_4^4\,\hat{b}_1^4\,\hat{b}_2^3\,\hat{b}_3^2.$$

We can also define $\hat{c}_k, \hat{f}_k, \hat{g}_k, \hat{h}_k, \beta_k, \cdots$ over the smooth locus of $\mathcal{S}pec\,\hat{R}$ as in §2 (with $R[\alpha]$ there replaced by $\hat{R}[y]$ here). By Theorem 2.1, A_0^* is given as follows (over the smooth locus of $\mathcal{S}pec\,\hat{R}$):

$$A_0^* = \left\{ \frac{\hat{u}_1\beta_1 + \hat{u}_2\beta_2 + \hat{u}_3\beta_3 + \hat{u}_4\beta_4}{\hat{c}_0} \;\middle|\; \hat{u}_j \in \hat{R},\ \hat{c}_0 \mid \hat{f}_k\hat{u}_k + \hat{g}_k\hat{u}_{k+1},\ k = 1, 2, 3 \right\}.$$

In the expression above, $\hat{c}_0|(\hat{f}_k\hat{u}_k + \hat{g}_k\hat{u}_{k+1})$ means that $\hat{f}_k\hat{u}_k + \hat{g}_k\hat{u}_{k+1} - \hat{c}_0\hat{u}_{k+4} = 0$ for some $\hat{u}_{k+4}$ in $\hat{R}$. Using bases, we have the expressions below, where u_k are in $\hat{R}_3$, v_k are in $\hat{R}_2$, w_k are in $\hat{R}_1$ and x_k are in R

$$\begin{aligned} \hat{u}_k &= u_k + u_{k+7}\gamma_1 + u_{k+14}\gamma_2 \ \ (1 \leq k \leq 7), \\ u_k &= v_k + v_{k+21}\sqrt{b_{11}c_{11}} \ \ (1 \leq k \leq 21), \\ v_k &= w_k + w_{k+42}\sqrt{\mu_{4,3}} \ \ (1 \leq k \leq 42), \\ w_k &= x_k + x_{k+84}\sqrt{\mu_{2,1}} \ \ (1 \leq k \leq 84). \end{aligned}$$

So we can write $\hat{u}_k = \sum_{j=1}^{168} \varepsilon_{kj} x_j$, where ε_{kj} is in the ring $\mathbf{Z}[\sqrt{\mu_{2,1}}, \sqrt{\mu_{4,3}}, \sqrt{b_{11}c_{11}}, \gamma_1, \gamma_2]$. Now we can express a general element x of A_0^* as follows

$$x = \frac{\hat{u}_1\beta_1 + \hat{u}_2\beta_2 + \hat{u}_3\beta_3 + \hat{u}_4\beta_4}{\hat{c}_0} = \sum_{j=1}^{168} x_j d_j,$$

$$d_j = \frac{\sum_{k=1}^{4} \varepsilon_{kj}\beta_k}{\hat{c}_0}.$$

We can calculate all d_j $(1 \le j \le 168)$ more explicitly as follows, where those d_j not listed below, are equal to 0

$$\begin{aligned}
\hat{c}_0 d_k &= \beta_k \ \ (1 \le k \le 4),\\
\hat{c}_0 d_k &= \gamma_1\beta_{k-7} \ \ (8 \le k \le 11),\\
\hat{c}_0 d_k &= \gamma_2\beta_{k-14} \ \ (15 \le k \le 18),\\
\hat{c}_0 d_k &= \sqrt{b_{11}c_{11}}\beta_{k-21} \ \ (22 \le k \le 25),\\
\hat{c}_0 d_k &= \gamma_1\sqrt{b_{11}c_{11}}\beta_{k-28} \ \ (29 \le k \le 32),\\
\hat{c}_0 d_k &= \gamma_2\sqrt{b_{11}c_{11}}\beta_{k-35} \ \ (36 \le k \le 39),\\
\hat{c}_0 d_k &= \sqrt{\mu_{4,3}}\beta_{k-42} \ \ (43 \le k \le 46),\\
\hat{c}_0 d_k &= \gamma_1\sqrt{\mu_{4,3}}\beta_{k-49} \ \ (50 \le k \le 53),\\
\hat{c}_0 d_k &= \gamma_2\sqrt{\mu_{4,3}}\beta_{k-56} \ \ (57 \le k \le 60),\\
\hat{c}_0 d_k &= \sqrt{b_{11}c_{11}\mu_{4,3}}\beta_{k-63} \ \ (64 \le k \le 67),\\
\hat{c}_0 d_k &= \gamma_1\sqrt{b_{11}c_{11}\mu_{4,3}}\beta_{k-70} \ \ (71 \le k \le 74),\\
\hat{c}_0 d_k &= \gamma_2\sqrt{b_{11}c_{11}\mu_{4,3}}\beta_{k-77} \ \ (78 \le k \le 81),\\
\hat{c}_0 d_k &= \sqrt{\mu_{2,1}}\beta_{k-84} \ \ (85 \le k \le 88),\\
\hat{c}_0 d_k &= \gamma_1\sqrt{\mu_{2,1}}\beta_{k-91} \ \ (92 \le k \le 95),\\
\hat{c}_0 d_k &= \gamma_2\sqrt{\mu_{2,1}}\beta_{k-98} \ \ (99 \le k \le 102),\\
\hat{c}_0 d_k &= \sqrt{b_{11}c_{11}\mu_{2,1}}\beta_{k-105} \ \ (106 \le k \le 109),\\
\hat{c}_0 d_k &= \gamma_1\sqrt{b_{11}c_{11}\mu_{2,1}}\beta_{k-112} \ \ (113 \le k \le 116),\\
\hat{c}_0 d_k &= \gamma_2\sqrt{b_{11}c_{11}\mu_{2,1}}\beta_{k-119} \ \ (120 \le k \le 123),\\
\hat{c}_0 d_k &= \sqrt{\mu_{4,3}\mu_{2,1}}\beta_{k-126} \ \ (127 \le k \le 130),\\
\hat{c}_0 d_k &= \gamma_1\sqrt{\mu_{4,3}\mu_{2,1}}\beta_{k-133} \ \ (134 \le k \le 137),\\
\hat{c}_0 d_k &= \gamma_2\sqrt{\mu_{4,3}\mu_{2,1}}\beta_{k-140} \ \ (141 \le k \le 144),\\
\hat{c}_0 d_k &= \sqrt{b_{11}c_{11}\mu_{4,3}\mu_{2,1}}\beta_{k-147} \ \ (148 \le k \le 151),\\
\hat{c}_0 d_k &= \gamma_1\sqrt{b_{11}c_{11}\mu_{4,3}\mu_{2,1}}\beta_{k-154} \ \ (155 \le k \le 158),\\
\hat{c}_0 d_k &= \gamma_2\sqrt{b_{11}c_{11}\mu_{4,3}\mu_{2,1}}\beta_{k-161} \ \ (162 \le k \le 165).
\end{aligned}$$

Now we look at the syzygy condition in A_0^*. We can write

$$\hat{f}_k\hat{u}_k + \hat{g}_k\hat{u}_{k+1} - \hat{c}_0\hat{u}_{k+4} = \sum_{j=1}^{168} x_j e_{kj},$$

$$e_{kj} = \hat{f}_k\varepsilon_{kj} + \hat{g}_k\varepsilon_{k+1,j} - \hat{c}_0\varepsilon_{k+4,j}.$$

Let $A := R[\alpha]$, let $\widetilde{A}$ be the normalization of A in the fraction field of A and let $\widetilde{A}_0$ be the R-submodule of $\widetilde{A}$ consisting of trace (over R) free elements. Clearly, $\widetilde{A}_0 = \{\mathrm{Tr}(x) \,|\, x \in A_0^*\}$ (see **(4.5)** and **(4.6) (2)** below for Tr). Thus we have proved the main theorem below of this section, since an element in a normal ring $\widetilde{A}$ is determined uniquely by its restriction to the complement of a co-dimension 2 subset.

Theorem 4.5. *Let R be a Noetherian* UFD *containing a field with* $\mathrm{Char}\, R$ *coprime to* $2, 3, 5$. *For a general degree-5 extension $R \subset R[\alpha]$, where α is a root of an irreducible polynomial $f(z) = z^5 + \sigma_2 z^3 - \sigma_3 z^2 + \sigma_4 z - \sigma_5$ in $R[z]$, the integral closure $\widetilde{A}$ of $R[\alpha]$ in the fraction field of $R[\alpha]$ is given by $\widetilde{A} = R \oplus \widetilde{A}_0$, and the trace (over R) free R-submodule $\widetilde{A}_0$ of $\widetilde{A}$ is given as follows:*

$$\widetilde{A}_0 = \left\{ \sum_{i=1}^{168} x_i \mathrm{Tr}(d_i) \mid x_i \in R, \ \sum_{j=1}^{168} x_j e_{kj} = 0, \quad (k = 1, 2, 3) \right\},$$

where Tr *is the trace for the field extension $Q(R[\alpha]) \subset Q(\hat{R}[\alpha])$.*

Remark 4.6. (1) In Theorem 4.5, by a general degree-5 extension, we mean that the conditions that $\sigma_2 \neq 0$ and $\tau_i \neq 0$ $(i = 1, 2)$ in **(4.2)** are satisfied and that the extension $R \subset \hat{R}$ is of degree 24. If the extension has a smaller degree, the same process of reducing to a B-J extension works and is even simpler by a least one step.

(2) For the calculation of $\mathrm{Tr}(d_i)$, we note that

$$\mathrm{Tr}|\hat{R}[\alpha]/R[\alpha] = \hat{T}_1 \circ \hat{T}_2 \circ \hat{T}_3 \circ \hat{T}_4,$$

and $\hat{T}_i = \mathrm{Tr}|\hat{R}_i[\alpha]/\hat{R}_{i-1}[\alpha]$ is the lifting of $T_i = \mathrm{Tr}|\hat{R}_i/\hat{R}_{i-1}$, where $\hat{R}_0 := R$ and $\hat{R}_4 := \hat{R}$. The traces T_i $(i = 1, 2, 3)$ for quadratic extensions are rather easy and for T_4 we have $T_4(u_0 + u_1\gamma_1 + u_2\gamma_2) = u_0$ whenever $u_i \in \hat{R}_3$

5. Some applications in algebraic geometry

We will now apply the calculation of integral closure to B-J covers of a factorial variety.

Let X be a factorial variety over a field k with $\mathrm{Char}\, k$ coprime to both $n-1$ and n, let L be a line bundle over X, and let s and t be two non-zero global sections of L^{n-1} and L^n, respectively. Then we can construct a normal finite cover $\pi : Y \to X$ of degree n by adding a root of the B-J polynomial $f = z^n + sz + t$, which is irreducible over the function field $k(X)$. We may also assume that the data (s, t) is minimal. The construction is as follows.

Let $p : [L] \to X$ be the $\mathbf{A}^1$-bundle over X associated with the line bundle L. Denote by z the fiber coordinate. Then $f = z^n + sz + t$ is a global section of $p^*(L^n)$. Denote by $\overline{Y}$ the zero scheme of f. Let Y be the normalization of $\overline{Y}$. Then we see that the induced cover $\pi : Y \to X$ is a finite morphism of degree n. We call it a B-J *cover.*

We have the same type of factorizations of s and t as in §1. So we can define global sections f_i, g_i and h_i .... of some line bundles as in §2. The globalization of Proposition 1.2 shows that there is a one to one correspondence between the B-J polynomials $z^n + sz + t$ with the data (s, t) minimal and the triplets (a, b, c) of coprime sections of a line bundle with $a + b = c$. So $a + b = c$ can be viewed as the covering data of a B-J cover.

We denote by E_π the trace free subsheaf of $\pi_* \mathcal{O}_Y$. Since $\mathrm{Char}\, k$ is coprime to n, the trace map $\mathrm{Tr} : \pi_* \mathcal{O}_Y \to \mathcal{O}_X$ splits $\pi_* \mathcal{O}_X$ as the direct sum of $\mathcal{O}_X$ and E_π:

$$\pi_* \mathcal{O}_Y = \mathcal{O}_X \oplus E_\pi.$$

Now we construct a $(n-2) \times (2n-3)$ matrix $M = (m_{ij})$:

$$m_{ij} := \begin{cases} f_i, & \text{if } \ j = i, \\ g_i, & \text{if } \ j = i+1, \\ c_0, & \text{if } \ j = i+n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Define $2n-3$ hypersurfaces ($g_0 := 1, \ f_{n-1} := 1$):

$$V_i := \begin{cases} \operatorname{div}(g_1 \cdots g_{i-1} f_i \cdots f_{n-2}), & \text{if } \ 1 \le i \le n-1, \\ \operatorname{div}(c_0 g_1 \cdots g_k f_{k+2} \cdots f_{n-2}), & \text{if } \ i = n+k, \ 0 \le k \le n-3. \end{cases}$$

$V_1, \cdots, V_{2n-3}$ determine a syzygy sheaf $\mathcal{F}$

$$0 \to \mathcal{F} \to \oplus_{i=1}^{2n-3} \mathcal{O}_X(-V_i) \xrightarrow{M} \oplus_{i=1}^{n-2} \mathcal{O}_X(-V_i + \operatorname{div}(f_i)),$$
$$(v_1, \cdots, v_{2n-3})^t \mapsto M(v_1, \cdots, v_{2n-3})^t.$$

We have chosen V_i so that the map M is well defined. Namely, the i-th component of an element in the image of M is a $\mathbf{Z}$-combination of local sections in the same line bundle $\mathcal{O}_X(-V_i + \operatorname{div}(f_i))$ with a fixed transition function. One can check easily that the divisor

$$T := V_i - iL + \operatorname{div}(c_0) + \operatorname{div}(h_i)$$

is independent of $i = 1, \ldots, n-1$. Now we state the main result of the section:

Theorem 5.1. *Let $\pi : Y \to X$ be a degree-n finite morphism from a normal variety onto the factorial variety X defined over a field k with* $\operatorname{Char} k$ *coprime to n and $n-1$, so that the function field $k(Y)$ is obtained by adding to $k(X)$ a root of a polynomial $f = z^n + sz + t$, where $0 \neq s \in H^0(X, L^{n-1})$, $t \in H^0(X, L^n)$ for a line bundle L with the data (s,t) minimal in the sense of* (1.1) (*for π, see the beginning of* §5 *for more detail*). *Then $\pi_* \mathcal{O}_Y = \mathcal{O}_X \oplus E_\pi$ and the trace* (*over X*) *free part $E_\pi \cong \mathcal{F}(T)$.*

Proof. We define a map

$$\tau : \mathcal{F}(T) \to E_\pi, \quad (v_1, \cdots, v_{2n-3})^t \mapsto \frac{\sum_{i=1}^{n-1} v_i \beta_i}{c_0}.$$

Since $\beta_i / c_0 = \alpha^i / h_i c_0$ can be viewed as a basis of $\mathcal{O}_X(T - V_i)$, we can check that the linear map is well defined globally. By Theorem 2.1, τ is an isomorphism locally and hence globally.

Note that even if X is only normal, the above argument works too. In fact, the above study works for the induced B-J cover over the smooth part of X. Since the singular locus of X is of co-dimension at least 2, most of the results can be extended to X.

If X is a smooth surface, then the singularities of any B-J cover $\pi : Y \to X$ can be resolved by the following classical method. Let $a + b = c$ be the data of π. If the branch locus has a singular point p_1 worse than normal crossing, then we blow up X at p_1: $\sigma_1 : X_1 \to X$. Let $\pi_1 : Y_1 \to X_1$ be the pullback of π via σ_1, i.e., Y_1 is the normalization of $X_1 \times_X Y$. This π_1 is a B-J cover defined by $z^n + \sigma_1^*(s) z + \sigma_1^*(t) = 0$. If we denote by $a^{(1)} + b^{(1)} = c^{(1)}$ the minimal data of π_1, then it is obtained from

$$\sigma_1^*(a) + \sigma_1^*(b) = \sigma_1^*(c)$$

by eliminating the common factors from both sides, which come from the exceptional divisor. After a finite number of steps, the branch locus of π_m becomes normal crossing. Now we can resolve the singularities of Y_m by the Hirzebruch-Jung method.

When $n = 3$, it has been proved in [Ta2] that the singularities of Y_m can be resolved by simply applying the same process as above to the singular points of the branch locus. Thus for any triple cover, we have a "canonical resolution" of the singularities the same as in the double cover case (see [Ta2, Theorem 7.2]).

The result below follows from Theorem 5.1, the fact that $(n-1)L \sim \mathrm{div}(s)$ and the fact that the cokernel of M in the exact sequence preceding Theorem 5.1 is a direct sum of sheaves each summand of which is supported on a set $\{f_i = g_i = c_0 = 0\}$ (which has codimension at least 2); see Cor. 2.2 for the local expression of M.

Corollary 5.2. *With the same assumptions as in Theorem* 5.1*, we have linear equivalences:*

$$c_1(\pi_*\mathcal{O}_Y) \sim (n-1)T - \sum_{i=n-1}^{2n-3} V_i - \sum_{i=1}^{n-2} \mathrm{div}(f_i) \sim \mathrm{div}\left(\frac{c_0}{s}\prod_{i=1}^{n-2}\left(\frac{f_i}{g_i}\right)^{n-1-i}\right).$$

For a divisor D on a variety, we denote by D_{red} the reduced divisor with the same support as D. For a morphism f we denote by R_f the ramification divisor and $B_f = f_*(R_f)$ the branch locus of f; when f is a finite morphism of degree n, we write $R_f = R_{fs} + R_{ft} + \cdots$, $B_{fs} := f_*R_{fs}$, $B_{ft} := f_*R_{ft}$, where f is *simply* (resp. *totally*) ramified along R_{fs} (resp. R_{ft}), i.e., over a generic point of $(B_{fs})_{\mathrm{red}}$ (resp. $(B_{ft})_{\mathrm{red}}$) the ramification index(es) are $2, 1, \ldots, 1$ (resp. is n). So $R_{ft} = (n-1)(R_{ft})_{\mathrm{red}}$, $B_{ft} = (n-1)(B_{ft})_{\mathrm{red}}$; R_{fs} and B_{fs} are reduced. When $n = 3$, we have $R_f = R_{fs} + R_{ft}$.

Theorem 5.3. *Let $\pi : Y \to X$ be a degree-3 finite morphism from a normal variety onto the factorial variety X defined over a field k with* $\mathrm{Char}\, k \neq 2, 3$*. Then we have:*

(1) *We have a linear equivalence:*

$$c_1(\pi_*\mathcal{O}_Y) \sim -(B_{\pi t})_{\mathrm{red}} - \eta,$$

where η is a divisor satisfying $2\eta \sim B_{\pi s}$.

(2)

$$2c_1(\pi_*\mathcal{O}_Y) \sim -B_\pi.$$

(3) *π is unramified outside a co-dimension two subset if and only if $2c_1(\pi_*\mathcal{O}_Y) \sim 0$.*

We now prove Theorem 5.3. Since $\mathrm{Char}\, k \neq 3$, we see that π is a B-J cover and given by a polynomial below (see [Ta2, Theorem 7.2]):

$$f(z) = z^3 + sz + t$$

where s, t are global sections of L^2, L^3, where L is a line bundle on X. Since $f(z)$ is defined globally, z (a zero of f), s, t have the transition functions $\sigma_{ij}, \sigma_{ij}^2, \sigma_{ij}^3$, respectively, with respect to affine open sets $\{U_i\}$ covering X. We may assume that the data (s, t) is minimal. Thus Theorem 5.3 follows from Theorem 1.4 and Lemma 5.4 below.

We will use the notations $A_i = \mathrm{div}(a_i), B_i = \mathrm{div}(b_i), C_i = \mathrm{div}(c_i)$; see §1 for a_k, b_k and c_k. Lemma 5.4, where (1) was proved in [Ta1, Ta2], is now a consequence of Corollary 5.2; in applying, we also use the fact that $\mathrm{div}(s^n/t^{n-1}) \sim 0$ and $C_1 + 2C_0 \sim B_1 + 2B_0$ from the definition of c_i (for **(5.4) (1)**).

Lemma 5.4. *We assume the hypothesis and notation in Theorem* 5.1 *or Corollary* 5.2.

(1) *Let $n = 3$, i.e., let $\pi : Y \to X$ be a degree* 3 *finite morphism from a normal variety onto the factorial variety X defined over a field k with* $\mathrm{Char}\, k \neq 2, 3$, *so that the function field $k(Y)$ is obtained by adding to $k(X)$ a root of a polynomial $f = z^3 + sz + t$, where $0 \neq s \in H^0(X, L^2)$, $t \in H^0(X, L^3)$ for a line bundle L with the data (s, t) minimal in the sense of* (1.1) (*for π, see the beginning of* §5 *for more detail*). *Then we have:*

$$c_1(\pi_*\mathcal{O}_Y) = C_0 - A_1 - A_2 - B_0 - B_1 = -(A_1 + A_2) - \eta,$$

where $\eta = B_0 + B_1 - C_0$ and $2\eta \sim B_1 + C_1$.

(2) *If $n = 4$, then we have the linear equivalence:*

$$c_1(\pi_*\mathcal{O}_Y) \sim \mathrm{div}\left(\frac{c_0}{a_0^2 a_1^2 a_2^2 a_3^3 b_1 b_2}\right).$$

(3) *If $n = 5$, then we have:*

$$c_1(\pi_*\mathcal{O}_Y) \sim \mathrm{div}\left(\frac{c_0}{(a_1 a_2 a_3 a_4 b_0 b_1 b_2)^2\, b_3^3}\right).$$

The first attempted proof of the following result appeared in [Ta2]. The approach here is different.

Theorem 5.5. *Let $\pi : Y \to X$ be a finite morphism of degree* 3 *from a normal variety onto a factorial variety defined over a field k with* $\mathrm{Char}\, k \neq 2, 3$. *Suppose that $H^0(X, \mathcal{O}_X) = k$ and every element of k has a square root in k. Then π is Galois if and only if the following two conditions are satisfied.*

(1) *Outside a co-dimension* 2 *subset, π is either unramified or totally ramified, i.e., either $B_\pi = 0$ or $\pi^*((B_\pi)_{\mathrm{red}}) = 3(R_\pi)_{\mathrm{red}}$.*
(2) $c_1(\pi_*\mathcal{O}_Y) \sim -(B_\pi)_{\mathrm{red}}$.

Proof. We may assume that π is given as in Lemma 5.4 [Ta2, Theorem 7.2], though s might be identically 0.

Suppose that π is Galois. Then (1) is clear and we may assume that $f = z^3 + \ell_1\ell_2^2$ with $\mathrm{div}(\ell_i)$ reduced. By Lemma 1.3, we have $\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X z \oplus \mathcal{O}_X(z^2/\ell_2)$. Hence we have:

$$c_1(\pi_*\mathcal{O}_Y) = -\mathrm{div}(z) - \mathrm{div}(z^2/\ell_2) = -\mathrm{div}(z^3/\ell_2) = -\mathrm{div}(\ell_1\ell_2^2/\ell_2) = -(B_\pi)_{\mathrm{red}}.$$

Now we assume (1) and (2). In notation of Lemma 5.4, π is Galois if $s = 0$. So we may assume that $s \neq 0$ and all conditions of Lemma 5.4 are satisfied. As in §1, we have the expression below, where a_k, b_k, c_k are global sections of some line bundles

$$s = a_0 a_1 a_2^2 b_1, \quad t = b_0 a_1 a_2^2 b_1^2.$$

Note that the discriminant δ of $f(z)$ is given by

$$\delta = (a_1 a_2^2)^2 b_1^3 c_1 c_0^2, \;\; c_1 c_0^2 = 4 a_1 a_2^2 a_0^3 + 27 b_1 b_0^2.$$

Write $a_k = \{a_k(i)\}, b_k = \{b_k(i)\}, c_k = \{c_k(i)\}$ where $a_k(i), \ldots$ are regular sections on the affine open set U_i, the union of which covers X. From the abstract algebra, we know that

when Char $k \neq 2$, the finite morphism $\pi : Y \to X$ is Galois if and only if δ has a square root in the function field $k(X)$, i.e., $b_1(i)c_1(i)$ has a square root in $k(X)$ for one i (and hence for all i).

Note that $B_1 = C_1 = 0$ for otherwise π would be of simple ramification over $B_1 = \mathrm{div}(b_1)$ and $C_1 = \mathrm{div}(c_1)$ (Theorem 1.4), contradicting (1). So $b_1(i)$ and $c_1(i)$ are invertible on U_i. We may assume that $b_1(i) = 1$ by renaming a_0b_1 and $b_0b_1^2$ as new a_0 and b_0, respectively so that $s = a_0a_1a_2^2$ and $t = b_0a_1a_2$. Note that for all i, we have

$$c_1(i)c_0(i)^2 = 4a_1(i)a_2(i)^2a_0(i)^3 + 27b_0(i)^2.$$

The 3 terms in the equation have the same transition function; indeed this equation is deduced by renaming the quotient of $\delta = 4s^3 + 27t^2$ after the division by the common factors of the two terms in δ, while the transition functions of s, t are $\sigma_{ij}^2, \sigma_{ij}^3$, respectively (so the two terms in δ have the same transition function).

On the other hand, from the condition (2) and Lemma 5.4, we see that $B_0 \sim C_0$. We may assume that $b_0 = \{b_0(i)\}$ and $c_0 = \{c_0(i)\}$ have the same transition function after adjusting c_1 if necessary. This and the similar fact after the display in the previous paragraph force $c_1 = \{c_1(i)\}$ to have the constant 1 as its transition function, i.e., $c_1(i) = c_1(j)$ for all i, j. So $c_1(i)$ is a global invertible function. Hence $c_1(i)$ is a constant in k because $H^0(X, \mathcal{O}_X) = k$. Now $b_1(i)c_1(i) = c_1(i)$ has a square root in $k \subset k(X)$. So the Galoisness follows. This proves the theorem.

Remark 5.6. (1) The condition (1) alone in Theorem 5.5 is not enough to imply the Galoisness of π. Indeed, by a Theorem of J. P. Serre, we know that for any $m \geq 2$ there is an m-dimensional projective complex manifold X with $\pi_1(X) = S_3$, the symmetric group in 3 letters. Let U be the universal cover of X and let $Y = U/\langle\sigma\rangle$ where σ is an involution in S_3. Then $Y \to X = U/S_3$ is a finite morphism of degree 3 between smooth projective manifolds, which is unramified but non-Galois.

(2) Despite what we said in (1), the condition (1) in Theorem 5.5 together with a condition (2)' that $PicX$ has no 2-torsion element (this is true when $\pi_1(X)$ has no index-2 subgroup) will imply the Galoisness of π. Indeed, the conditions (1) and (2)' imply that $\eta = 0$ in Lemma 5.4 and hence the condition (2) in Theorem 5.5 holds.

(3) We like to have a similar geometric Galoisness criterion for degree 4, or 5 extension, but we realized from the discussion with Professor Catanese that it is much more complicated. The following is a partial result.

(4) In Theorem 5.5 and Proposition 5.7(2), reading the proof, we see that the two conditions that $H^0(X, \mathcal{O}_X) = k$ and every element in k has a square root in k can be weakened to one condition that every element of $H^0(X, \mathcal{O}_X)$ has a square root in the function field $k(X)$.

Proposition 5.7. *With the assumption in Theorem* 5.1*, we set* $n = 5$*. Then we have:*

(1) *If* π *is Galois, i.e., if the Galois group* $\mathrm{Gal}(f)$ *of the polynomial* $f(z)$ *is* $\mathbf{Z}/(5)$*, then the following are true:*

- (1a) $c_1(\pi_*\mathcal{O}_Y) \sim -2(B_\pi)_{\mathrm{red}}$*, and*
- (1b) *Outside a co-dimension* 2 *subset,* π *is either unramifield or totally ramified, i.e., either* $B_\pi = 0$ *or* $\pi^*((B_\pi)_{\mathrm{red}}) = 5(R_\pi)_{\mathrm{red}}$.

(2) *Conversely, suppose that* (1a) *and* (1b) *are satisfied and suppose further that* $H^0(X, \mathcal{O}_X) = k$ *and every element in the ground field* k *has a square root in* k*. Then* $\mathrm{Gal}(f)$ *is one of* $\mathbf{Z}/(5)$*,* D_{10} (*the dihedral group of order* 10) *and* A_5 (*the alternating group in* 5 *letters*).

Proof. Assume first that π is Galois. Then (1b) is clear. We may also assume that the function field $k(Y)$ is obtained by adding to $k(X)$ a root of the polynomial below

$$z^5 + \ell_1\ell_2^2\ell_2^3\ell_4^4$$

where ℓ_j are reduced global sections of some line bundles. Then, by Lemma 1.3, we have:

$$\pi_* \mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X z \oplus \mathcal{O}_X(z^2/(\ell_3\ell_4)) \oplus \mathcal{O}_X(z^3/(\ell_2\ell_3\ell_4^2)) \oplus \mathcal{O}_X(z^4/(\ell_2\ell_3^2\ell_4^3)).$$

As in Theorem 5.5, we can now imply $(1a)$:

$$c_1(\pi_*\mathcal{O}_Y) = \mathrm{div}(\ell_2^2\ell_3^4\ell_4^6/z^{10}) \sim -2\mathrm{div}(\ell_1\ell_2\ell_3\ell_4) = -2(B_\pi)_{\mathrm{red}}.$$

Here we used the fact that $\mathrm{div}(z^{10}) \sim \mathrm{div}(\ell_1\ell_2^2\ell_2^3\ell_4^4)^2$.

Now assume $(1a)$ and $(1b)$. So $C_1 = \mathrm{div}(c_1) = 0$ and $B_i = \mathrm{div}(b_i) = 0$ for all $i \ge 1$ (see Theorem 1.4). By Lemma 5.4, we have $c_1(\pi_*\mathcal{O}_Y) \sim C_0 - 2(A_1 + A_2 + A_3 + A_4 + B_0)$. This and $(1a)$ imply that $C_0 \sim 2B_0$. The rest is similar to Theorem 5.5. To be precise, since $B_i = \mathrm{div}(b_i) = 0$ $(i \ge 1)$ we can adjust a_0 and b_0 and assume that $b_i = 1$ $(i \ge 1)$ so that $s = a_0a_1a_2^2a_3^3a_4^4$ and $t = b_0a_1a_2^2a_3^3a_4^4$. So the discriminant δ of the polynomial $f(z)$ in Theorem 5.1 equals $(a_1a_2^2a_3^3a_4^4)^4c_1^2c_0$ where

$$c_1c_0^2 = 256a_1a_2^2a_3^3a_4^4a_0^5 + 3125b_0^4.$$

The 3 terms in the equation above have the same transition function; since $C_0 \sim 2B_0$, we can adjust c_1 and assume that $C_0 = \mathrm{div}(c_0)$ and $2B_0 = \mathrm{div}(b_0^2)$ have the same transition function. These two facts imply that c_1 has constant 1 as its transition function, so c_1 is a non-zero scalar in k. Thus the discriminant δ of $f(z)$ has a square root in $k(X)$, whence $\mathrm{Gal}(f) \le A_5$ (noting that $\mathrm{Char}\, k \ne 2$ for it is coprime to $5-1$ (and also 5)). Now the assertion (2) follows from the classification of subgroups in A_5.

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